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Well-posedness of dual-phase-lagging heat conduction equation: higher dimensions

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Abstract

The dual-phase-lagging heat conduction equation is shown to be of one unique solution for a finite region of dimension n ($n \geqslant 2$) under Dirichlet, Neumann or Robin boundary conditions. The solution is also found to be stable with respect to initial conditions. The work is of fundamental importance in applying the dual-phase-lagging model for the microscale heat conduction of high-rate heat flux. © 2002 Published by Elsevier Science Ltd.

1. Introduction

The dual-phase-lagging heat conduction equation originates from the first law of thermodynamics and the dual-phase-lagging constitutive relation of heat flux density [1,2]. It is developed in examining energy transport involving the high-rate heating in which the non-equilibrium thermodynamic transition and the microstructural effect become important associated with shortening of the response time. The high-rate heating is developing rapidly due to the advancement of high-power short-pulse laser technologies [3–7]. In addition to its application in the ultrafast pulse-laser heating, the dual-phase-lagging heat conduction equation also arises in describing and predicting phenomena such as temperature pulses propagating in superfluid liquid helium, nonhomogeneous lagging response in porous media, thermal lagging in amorphous materials, and effects of material defects and thermomechanical coupling [1]. Furthermore, the dual-phase-lagging heat conduction equation forms a generalized, unified equation with the classical parabolic heat conduction equation, the hyperbolic heat conduction equation, the energy equation in the phonon scattering model [8,9], and the energy equation in the phonon–electron interaction model [10–12] as its special cases [1,2]. This, with the rapid growth of microscale heat conduction of high-rate heat flux, has attracted the recent research effort on dual-phase-lagging heat conduction equations.

Solutions of one-dimensional (1D) heat conduction under some specific initial and boundary conditions were developed in [1,13–18]. The thermal oscillation and resonance described by the dual-phase-lagging heat conduction equations were examined in [19]. Conditions and features of thermal resonance, underdamped, critically damped and overdamped oscillations were obtained and compared with those described by the classical parabolic heat conduction equation and the hyperbolic heat conduction equation [19]. Wang and Zhou [2] developed methods of measuring the phase-lag of the heat flux and the temperature gradient and obtained analytical solutions for regular 1D, 2D and 3D heat conduction domains under essentially arbitrary initial and boundary conditions. Two solution structure theorems were also developed for dual-phase-lagging heat conduction equations under linear boundary conditions [2,20]. These theorems express contributions (to the temperature field) of the initial temperature distribution and the source term by that of the initial time-rate change of the temperature. They reveal the structure of the temperature field and considerably simplify the development of solutions of dual-phase-lagging heat conduction equations.

On the fundamental side, the dual-phase-lagging heat conduction equation was shown to be both admissible within the framework of the second law of the extended irreversible thermodynamics [1] and well-posed in a

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Nomenclature

h, k	nonnegative real constants
S	area
T	temperature
t	time
$\vec{x}$	position vector
V	Volume
Greek symbols	
α	thermal diffusivity
Ω	space domain
$\partial\Omega$	boundary of Ω
ϕ, ψ	known functions
τ_q	phase-lag of heat flux vector
τ_T	phase-lag of temperature gradient
∇	gradient
Δ	Laplacian
$\forall$	for all

finite 1D region under Dirichlet, Neumann or Robin boundary conditions [20]. The well-posed problem has been, however, left unaddressed for the n-dimensional case with $n \geqslant 2$. Furthermore, the method in [20] cannot be directly extended to the higher-dimension case. This stimulates the present work of modifying the method in [20] to examine the well-posedness of n-dimensional dual-phase-lagging heat conduction equations. In particular, we establish the existence, uniqueness and stability of the solution with respect to initial conditions for the n-dimensional dual-phase-lagging heat conduction with $n \geqslant 2$ under homogeneous Dirichlet, Neumann or Robin boundary conditions.

2. Existence

Consider the initial-boundary value problem of the n-dimensional ($n \geqslant 2$) dual-phase-lagging heat conduction equation under homogeneous Dirichlet, Neumann or Robin boundary conditions. [1]

$$\begin{aligned}
&\frac{1}{\alpha}T_t(\vec{x},t) + \frac{\tau_q}{\alpha}T_{tt}(\vec{x},t) \\
&\quad = \Delta T(\vec{x},t) + \tau_T\frac{\partial}{\partial t}\Delta T(\vec{x},t), \quad \Omega\times(0,+\infty), \\
&k\frac{\partial T(\vec{x},t)}{\partial n} + hT(\vec{x},t)|_{\partial\Omega} = 0, \quad (0,+\infty), \\
&T(\vec{x},0) = \phi(\vec{x},t), T_t(\vec{x},0) = \psi(\vec{x}), \quad \Omega.
\end{aligned} \tag{1}$$

Here t is the time, T can be the temperature or one component of heat flux vector, α is the thermal diffusivity, τ_T and τ_q are the phase-lags of the temperature gradient and the heat flux, respectively, $\vec{x}$ denotes a point in the space domain Ω of n-dimensions with the boundary $\partial\Omega$, Δ the Laplacian, ϕ and ψ are the known functions, $\partial T/\partial n$ the normal derivative of T, $T_t = \partial T/\partial t$, $T_{tt} = \partial^2 T/\partial t^2$, k and h are nonnegative real constants and satisfy $k + h \neq 0$.

Note that (1) covers the Dirichlet, Neumann and Robin boundary conditions by different combinations of k and h. The readers are referred to [1,21] for physical implications and limitations of three boundary conditions.

Assume separation of the variables in the form

$$T(\vec{x},t) = X(\vec{x})\Gamma(t). \tag{2}$$

A substitution of (2) into (1) yields

$$X(\vec{x})\left[\frac{1}{\alpha}\Gamma_t(t) + \frac{\tau_q}{\alpha}\Gamma_{tt}(t)\right] = \Delta X(\vec{x})[\Gamma(t) + \tau_T\Gamma_t(t)],$$

which becomes, after dividing by $X(\vec{x})[\Gamma(t) + \tau_T\Gamma_t(t)]$, [2]

$$\frac{\frac{1}{\alpha}\Gamma_t(t) + \frac{\tau_q}{\alpha}\Gamma_{tt}(t)}{\Gamma(t) + \tau_T\Gamma_t(t)} = \frac{\Delta X(\vec{x})}{X(\vec{x})}. \tag{3}$$

We have, therefore, the separation equation for the temporal variable $\Gamma_t(t)$

$$\tau_q\Gamma_{tt}(t) + (1 + \alpha\tau_T\lambda)\Gamma_t(t) + \alpha\lambda\Gamma(t) = 0, \tag{4}$$

and the homogeneous system for the spatial variable $X(\vec{x})$

$$\Delta X(\vec{x}) + \lambda X(\vec{x}) = 0, \quad \vec{x} \in \Omega, \tag{5}$$

$$k\frac{\partial X}{\partial n} + hX = 0, \quad \vec{x} \in \partial\Omega, \tag{6}$$

where λ is the separation constant.

[1] For the heat conduction involving heat flux-specified boundary conditions, it is more convenient to use the dual-phase-lagging heat conduction equation in terms of the heat flux **q** or the heat flux potential ϕ defined by $\mathbf{q} = \nabla\phi$. The ϕ-version heat conduction equation has exactly the same structure as its T-version. The 1D **q**-version heat conduction equation is also of the same structure as its T-version. A mixed formulation for both **q** and T directly by two coupled energy and constitutive equations is more general in view of applications of the dual-phase-lagging model. The readers are referred to [1, pp. 30–34] for details.

[2] Note that $X(\vec{x})[\Gamma(t) + \tau_T\Gamma_t(t)]$ cannot be vanished for a nontrivial solution, see [20].

Integrating Eq. (5) over Ω after multiplying $X(\vec{x})$ leads to

$$\int_\Omega X\Delta X \, \mathrm{d}\Omega + \lambda \int_\Omega X^2 \, \mathrm{d}\Omega = 0 \tag{7}$$

which becomes, by applying the Green identity to $\int_\Omega X\Delta X \, \mathrm{d}\Omega$,

$$\int_{\partial\Omega} X \frac{\partial X}{\partial n} \, \mathrm{d}(\partial\Omega) - \int_\Omega \nabla X \cdot \nabla X \, \mathrm{d}\Omega + \lambda \int_\Omega X^2 \, \mathrm{d}\Omega = 0. \tag{8}$$

By Eq. (6), if $k \neq 0$, we obtain

$$\left.\frac{\partial X}{\partial n}\right|_{\partial\Omega} = -\frac{h}{k} X\bigg|_{\partial\Omega}.$$

Therefore, Eq. (8) becomes

$$\lambda \int_\Omega X^2 \, \mathrm{d}\Omega = \int_\Omega \nabla X \cdot \nabla X \, \mathrm{d}\Omega + \frac{h}{k} \int_{\partial\Omega} X^2 \, \mathrm{d}(\partial\Omega) \geqslant 0,$$

which implies

$$\lambda \geqslant 0. \tag{9}$$

For the case of $k = 0$, a similar analysis also leads to Eq. (9).

Using the solution of Eqs. (4)–(6) available in [2,20] for various Ω, we have a solution of Eq. (1):

$$\begin{aligned} T(\vec{x}, t) &= \sum_{m=1}^{\infty} \mathrm{e}^{v_m t} [A_m \cos \mu_m t + B_m \underline{\sin}(\mu_m t)] X_m(\vec{x}), \\ A_m &= \frac{1}{M_m} \int_\Omega \phi(\vec{x}) X_m(\vec{x}) \, \mathrm{d}\Omega, \\ B_m &= \frac{1}{M_m \underline{\mu_m}} \int_\Omega \psi(\vec{x}) X(\vec{x}) \, \mathrm{d}\Omega \\ &\quad - \frac{v_m}{M_m \underline{\mu_m}} \int_\Omega \phi(\vec{x}) X_m(\vec{x}) \, \mathrm{d}\Omega, \end{aligned} \tag{10}$$

where

$$v_m = -\frac{1 + \alpha\tau_T \lambda_m}{2\tau_q}, \tag{11}$$

$$\mu_m = \frac{\sqrt{4\alpha\tau_q\lambda_m - (1 + \alpha\tau_T\lambda_m)^2}}{2\tau_q}, \tag{12}$$

$$\underline{\mu}_m = \begin{cases} \mu_m & \text{if } \mu_m \neq 0, \\ 1 & \text{if } \mu_m = 0, \end{cases} \tag{13}$$

$$\underline{\sin}(\mu_m t) = \begin{cases} \sin(\mu_m t) & \text{if } \mu_m \neq 0, \\ t & \text{if } \mu_m = 0, \end{cases} \tag{14}$$

and

$$M_m = \int_\Omega X_m^2(\vec{x}) \, \mathrm{d}\Omega. \tag{15}$$

Here λ_m and $X_m(\vec{x})$ are the eigenvalues and eigenfunctions of Eqs. (5) and (6), respectively. They are Ω-dependent and available in [2] for various Ω. Therefore, Eq. (1) has at least one solution.

3. Inequality

For the uniqueness and stability, we need first to develop an important inequality for (1). Note that

$$\begin{aligned} \frac{\partial}{\partial t}(T + \tau_q T_t)^2 &= 2(T + \tau_q T_t)(T_t + \tau_q T_{tt}) \\ &= 2\alpha(T + \tau_q T_t)(\Delta T + \tau_T \Delta T_t), \end{aligned} \tag{16}$$

in which the heat conduction equation in (1) has been used. Integrating (16) with respect to $\vec{x}$ over Ω yields

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial t}(T + \tau_q T_t)^2 \, \mathrm{d}\Omega \\ &\quad = 2\alpha \int_\Omega (T\Delta T + \tau_T T \Delta T_t + \tau_q T_t \Delta T + \tau_q \tau_T T_t \Delta T_t) \, \mathrm{d}\Omega. \end{aligned} \tag{17}$$

By the Green identity,

$$\begin{aligned} \int_\Omega T\Delta T \, \mathrm{d}\Omega &= \int_{\partial\Omega} T \frac{\partial T}{\partial n} \, \mathrm{d}(\partial\Omega) - \int_\Omega \nabla T \cdot \nabla T \, \mathrm{d}\Omega, \\ \int_\Omega T\Delta T_t \, \mathrm{d}\Omega &= \int_{\partial\Omega} T \frac{\partial T_t}{\partial n} \, \mathrm{d}(\partial\Omega) - \int_\Omega \nabla T \cdot \nabla T_t \, \mathrm{d}\Omega, \\ \int_\Omega T_t \Delta T \, \mathrm{d}\Omega &= \int_{\partial\Omega} T_t \frac{\partial T}{\partial n} \, \mathrm{d}(\partial\Omega) - \int_\Omega \nabla T_t \cdot \nabla T \, \mathrm{d}\Omega, \\ \int_\Omega T_t \Delta T_t \, \mathrm{d}\Omega &= \int_{\partial\Omega} T_t \frac{\partial T_t}{\partial n} \, \mathrm{d}(\partial\Omega) - \int_\Omega \nabla T_t \cdot \nabla T_t \, \mathrm{d}\Omega. \end{aligned}$$

Also note that,

$$\begin{aligned} \int_\Omega \nabla T \cdot \nabla T_t \, \mathrm{d}\Omega &= \int_\Omega \nabla T_t \cdot \nabla T \, \mathrm{d}\Omega \\ &= \frac{1}{2} \int_\Omega \frac{\partial}{\partial t}(\nabla T \cdot \nabla T) \, \mathrm{d}\Omega. \end{aligned}$$

Therefore, Eq. (17) can be rewritten as

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial t}(T + \tau_q T_t)^2 \, \mathrm{d}\Omega + \alpha(\tau_T + \tau_q) \int_\Omega \frac{\partial}{\partial t}(\nabla T \cdot \nabla T) \, \mathrm{d}\Omega \\ &\quad = 2\alpha \int_{\partial\Omega} T \frac{\partial T}{\partial n} \, \mathrm{d}(\partial\Omega) - 2\alpha \int_\Omega \nabla T \cdot \nabla T \, \mathrm{d}\Omega \\ &\qquad + 2\alpha\tau_T \int_{\partial\Omega} T \frac{\partial T_t}{\partial n} \, \mathrm{d}(\partial\Omega) + 2\alpha\tau_q \int_{\partial\Omega} T_t \frac{\partial T}{\partial n} \, \mathrm{d}(\partial\Omega) \\ &\qquad + 2\alpha\tau_q\tau_T \int_{\partial\Omega} T_t \frac{\partial T_t}{\partial n} \, \mathrm{d}(\partial\Omega) - 2\alpha\tau_q\tau_T \int_\Omega \nabla T_t \cdot \nabla T_t \, \mathrm{d}\Omega. \end{aligned} \tag{18}$$

For the case of $k \neq 0$, the boundary condition in (1) becomes

$$\left.\frac{\partial T}{\partial n}\right|_{\partial\Omega} = -\frac{h}{k}T\bigg|_{\partial\Omega}. \tag{19}$$

Substituting Eq. (19) into Eq. (18) and defining

$$g(t) = \alpha(\tau_T + \tau_q)\int_\Omega \nabla T \cdot \nabla T \,\mathrm{d}\Omega + \frac{\alpha h}{k}(\tau_T + \tau_q) \times \int_{\partial\Omega} T^2 \,\mathrm{d}(\partial\Omega) + \int_\Omega (T + \tau_q T_t)^2 \,\mathrm{d}\Omega, \tag{20}$$

yield

$$\frac{\partial g(t)}{\partial t} = -\frac{2\alpha h}{k}\int_{\partial\Omega} T^2 \,\mathrm{d}(\partial\Omega) - 2\alpha\int_\Omega \nabla T\cdot\nabla T\,\mathrm{d}\Omega - 2\alpha\tau_q\tau_T\frac{h}{k}\int_{\partial\Omega} T_t^2\,\mathrm{d}(\partial\Omega) - 2\alpha\tau_q\tau_T\int_\Omega \nabla T_t\cdot\nabla T_t\,\mathrm{d}\Omega \tag{21}$$

which is negative semi-definite because α, τ_T, τ_q, h and k are all not negative. Therefore,

$$g(t_1) \leqslant g(t_0) \quad \forall t_1 \geqslant t_0. \tag{22}$$

The inequality for the case of $k = 0$ can also be written in the form of (22). However, the definition of $g(t)$ should be

$$g(t) = \alpha(\tau_T + \tau_q)\int_\Omega \nabla T\cdot\nabla T\,\mathrm{d}\Omega + \int_\Omega (T + \tau_q T_t)^2\,\mathrm{d}\Omega. \tag{23}$$

4. Uniqueness

Suppose that $T_1(\vec{x}, t)$ and $T_2(\vec{x}, t)$ are two solutions of (1). The difference between them

$$w(\vec{x}, t) = T_1(\vec{x}, t) - T_2(\vec{x}, t)$$

must satisfy the following initial-boundary value problem:

$$\begin{aligned} &\frac{1}{\alpha}w_t + \frac{\tau_q}{\alpha}w_{tt} = \Delta w + \tau_T\frac{\partial}{\partial t}\Delta w, \quad \Omega\times(0,\infty),\\ &k\frac{\partial w}{\partial n} + hw|_{\partial\Omega} = 0,\\ &w|_{t=0} = 0, \quad w_t|_{t=0} = 0. \end{aligned} \tag{24}$$

For the case of $k \neq 0$, an application of (22) to (24) yields, with $t_1 = t > 0$ and $t_0 = 0$,

$$\begin{aligned} &\alpha(\tau_T + \tau_q)\int_\Omega \nabla w(\vec{x}, t)\cdot\nabla w(\vec{x}, t)\,\mathrm{d}\Omega + \frac{\alpha h}{k}(\tau_T + \tau_q)\int_{\partial\Omega} w^2(\vec{x}, t)\,\mathrm{d}(\partial\Omega) + \int_\Omega [w(\vec{x}, t) + \tau_q w_t(\vec{x}, t)]^2\,\mathrm{d}\Omega\\ &\leqslant \alpha(\tau_T + \tau_q)\int_\Omega \nabla w(\vec{x}, 0)\cdot\nabla w(\vec{x}, 0)\,\mathrm{d}\Omega + \frac{\alpha h}{k}(\tau_T + \tau_q)\int_{\partial\Omega} w^2(\vec{x}, 0)\,\mathrm{d}(\partial\Omega) + \int_\Omega [w(\vec{x}, 0) + \tau_q w_t(\vec{x}, 0)]^2\,\mathrm{d}\Omega\\ &= 0. \end{aligned} \tag{25}$$

This requires

$$\nabla w(\vec{x}, t) = 0 \tag{26}$$

and

$$w(\vec{x}, t) + \tau_q w_t(\vec{x}, t) = 0. \tag{27}$$

Therefore, w is independent of $\vec{x}$ (Eq. (26)). The general solution of (27) is thus

$$w(\vec{x}, t) = c\,\mathrm{e}^{-t/\tau_q} \tag{28}$$

with c as a constant. Applying the initial condition $w(\vec{x}, 0) = 0$ yields

$$c = 0. \tag{29}$$

Therefore,

$$w(\vec{x}, t) = 0, \tag{30}$$

i.e.,

$$T_1(\vec{x}, t) = T_2(\vec{x}, t). \tag{31}$$

However, T_1 and T_2 are any two solutions of (1) so that the solution of (1) is unique. Similarly, we can also establish the uniqueness for the case of $k = 0$.

5. Stability

We establish the stability with respect to the initial conditions in the following stability theorem:

Stability Theorem. *If*

$$|\phi(\vec{x})| \leqslant \varepsilon, \tag{32}$$

$$|\psi(\vec{x})| \leqslant \varepsilon, \tag{33}$$

and

$$|\nabla\phi(\vec{x})| \leqslant \varepsilon, \tag{34}$$

the solution $T(\vec{x}, t)$ of (1) *satisfies*

$$|T(\vec{x}, t)| \leqslant c\varepsilon. \tag{35}$$

Here ε is a small positive constant, and c is a nonnegative constant.

Proof. For the case of $k \neq 0$, (22) yields, for (1) when $t > 0$ and $t_0 = 0$,

$$\begin{aligned}
&\alpha(\tau_T + \tau_q)\int_\Omega \nabla T \cdot \nabla T \, \mathrm{d}\Omega + \frac{\alpha h}{k}(\tau_T + \tau_q) \\
&\quad \times \int_{\partial\Omega} T^2 \, \mathrm{d}(\partial\Omega) + \int_\Omega (T + \tau_q T_t)^2 \, \mathrm{d}\Omega \\
&\quad \leqslant \alpha(\tau_T + \tau_q)\int_\Omega |\nabla\phi|^2 \, \mathrm{d}\Omega + \frac{\alpha h}{k}(\tau_T + \tau_q)\int_{\partial\Omega} \phi^2 \, \mathrm{d}(\partial\Omega) \\
&\qquad + \int_\Omega (\phi + \tau_q \psi)^2 \, \mathrm{d}\Omega \\
&\quad \leqslant \alpha(\tau_T + \tau_q)\varepsilon^2 V + \frac{\alpha h}{k}(\tau_T + \tau_q)\varepsilon^2 S + (1 + \tau_q)^2 \varepsilon^2 V \\
&\quad = \left[\left(\alpha V + \frac{\alpha h}{k} S\right)(\tau_T + \tau_q) + (1 + \tau_q)^2 V\right]\varepsilon^2 \\
&\quad = M\varepsilon^2, \qquad (36)
\end{aligned}$$

where Eqs. (32)–(34) have been used,

$$M = \left(\alpha V + \frac{2\alpha h}{k} S\right)(\tau_T + \tau_q) + (1 + \tau_q)^2 V, \qquad (37)$$

V is the volume of Ω, and S the area of $\partial\Omega$. Eq. (36) implies

$$\alpha(\tau_T + \tau_q)\int_\Omega |\nabla T|^2 \, \mathrm{d}\Omega \leqslant M\varepsilon^2, \qquad (38)$$

$$\frac{\alpha h(\tau_T + \tau_q)}{k}\int_{\partial\Omega} T^2 \, \mathrm{d}(\partial\Omega) \leqslant M\varepsilon^2, \qquad (39)$$

which are equivalent to

$$\int_\Omega |\nabla T|^2 \, \mathrm{d}\Omega \leqslant M_1\varepsilon^2, \qquad (40)$$

$$\int_{\partial\Omega} T^2 \, \mathrm{d}(\partial\Omega) \leqslant M_2\varepsilon^2. \qquad (41)$$

Here

$$M_1 = \frac{M}{\alpha(\tau_T + \tau_q)}, \quad M_2 = \frac{Mk}{\alpha h(\tau_T + \tau_q)}.$$

Since, for any two square-integrable functions f_1 and f_2,

$$\int_\Omega |f_1||f_2| \, \mathrm{d}\Omega \leqslant \sqrt{\int_\Omega f_1^2 \, \mathrm{d}\Omega}\sqrt{\int_\Omega f_2^2 \, \mathrm{d}\Omega},$$

$$\int_{\partial\Omega} |f_1||f_2| \, \mathrm{d}(\partial\Omega) \leqslant \sqrt{\int_{\partial\Omega} f_1^2 \, \mathrm{d}(\partial\Omega)}\sqrt{\int_{\partial\Omega} f_2^2 \, \mathrm{d}(\partial\Omega)},$$

we have

$$\int_\Omega |\nabla T| \, \mathrm{d}\Omega \leqslant \sqrt{\int_\Omega |\nabla T|^2 \, \mathrm{d}\Omega}\sqrt{\int_\Omega \mathrm{d}\Omega} \leqslant \sqrt{M_1 V}\varepsilon \qquad (42)$$

and

$$\int_{\partial\Omega} |T| \, \mathrm{d}(\partial\Omega) \leqslant \sqrt{\int_{\partial\Omega} T^2 \, \mathrm{d}(\partial\Omega)}\sqrt{\int_{\partial\Omega} \mathrm{d}(\partial\Omega)} \leqslant \sqrt{M_2 S}\varepsilon, \qquad (43)$$

in which (40) and (41) have been used. Eqs. (42) and (43) can be rewritten as

$$\int_\Omega \eta_1(\vec{x}, t) \, \mathrm{d}\Omega \leqslant 0 \quad \forall\Omega \qquad (44)$$

and

$$\int_{\partial\Omega} \eta_2(\vec{x}, t) \, \mathrm{d}(\partial\Omega) \leqslant 0 \quad \forall\partial\Omega, \qquad (45)$$

where

$$\eta_1(\vec{x}, t) = \left|\nabla T(\vec{x}, t)\right| - \sqrt{\frac{M_1}{V}}\varepsilon,$$

$$\eta_2(\vec{x}, t) = \left|T(\vec{x}, t)\right| - \sqrt{\frac{M_2}{S}}\varepsilon.$$

To obtain a local form of Eqs. (44) and (45), consider a subregion $\Omega_{\Delta v}$ of volume Δv around point $\vec{x_1}$ in Ω at arbitrary time instant τ, and a subsurface $\partial\Omega_{\Delta s}$ of area Δs around point $\vec{x_2}$ on $\partial\Omega$ at arbitrary time instant τ. Let

$$I_{\Delta v} = \left|\eta_1(\vec{x_1}, t) - \frac{1}{\Delta v}\int_{\Omega_{\Delta v}} \eta_1(\vec{x}, t) \, \mathrm{d}\Omega_{\Delta v}\right|, \qquad (46)$$

$$I_{\Delta s} = \left|\eta_2(\vec{x_2}, t) - \frac{1}{\Delta s}\int_{\partial\Omega_{\Delta s}} \eta_2(\vec{x}, t) \, \mathrm{d}\partial\Omega_{\Delta s}\right|. \qquad (47)$$

Then,

$$I_{\Delta v} \geqslant 0, \qquad (48)$$

$$I_{\Delta s} \geqslant 0. \qquad (49)$$

Also,

$$\begin{aligned}
I_{\Delta v} &= \left|\frac{1}{\Delta v}\int_{\Omega_{\Delta v}} [\eta_1(\vec{x_1}, t) - \eta_1(\vec{x}, t)] \, \mathrm{d}\Omega_{\Delta v}\right| \\
&\leqslant \frac{1}{\Delta v}\int_{\Omega_{\Delta v}} \max_{\vec{x} \in \Omega_{\Delta v}} \left|[\eta_1(\vec{x_1}, t) - \eta_1(\vec{x}, t)]\right| \, \mathrm{d}\Omega_{\Delta v} \\
&= \max_{\vec{x} \in \Omega_{\Delta v}} \left|[\eta_1(\vec{x_1}, t) - \eta_1(\vec{x}, t)]\right|,
\end{aligned}$$

$$\begin{aligned}
I_{\Delta s} &= \left|\frac{1}{\Delta s}\int_{\partial\Omega_{\Delta s}} [\eta_2(\vec{x_2}, t) - \eta_2(\vec{x}, t)] \, \mathrm{d}\partial\Omega_{\Delta s}\right| \\
&\leqslant \frac{1}{\Delta s}\int_{\Omega_{\Delta s}} \max_{\vec{x} \in \partial\Omega_{\Delta s}} \left|[\eta_2(\vec{x_2}, t) - \eta_2(\vec{x}, t)]\right| \, \mathrm{d}\partial\Omega_{\Delta s} \\
&= \max_{\vec{x} \in \partial\Omega_{\Delta v}} \left|[\eta_2(\vec{x_2}, t) - \eta_2(\vec{x}, t)]\right|,
\end{aligned}$$

which tend to zero as $\Delta v \to 0$ or $\Delta s \to 0$. Hence we have

$$I_{\Delta v} \leqslant 0 \quad \text{as } \Delta v \to 0, \qquad (50)$$

$$I_{\Delta s} \leqslant 0 \quad \text{as } \Delta s \to 0. \qquad (51)$$

To satisfy Eqs. (48)–(51), we have that

$$\eta_1(\vec{x_1}, \tau) = \lim_{\Delta v \to 0} \frac{1}{\Delta v}\int_{\Omega_{\Delta v}} \eta_1(\vec{x}, t) \, \mathrm{d}\Omega_{\Delta v} \qquad (52)$$

and

$$\eta_2(\vec{x_2},\tau) = \lim_{\Delta s\to 0}\frac{1}{\Delta s}\int_{\partial\Omega_{\Delta s}}\eta_2(\vec{x},t)\,\mathrm{d}\partial\Omega_{\Delta s}, \tag{53}$$

which yield, after using Eqs. (44) and (45),

$$\eta_1(\vec{x_1},\tau)\leqslant 0, \tag{54}$$

$$\eta_2(\vec{x_2},\tau)\leqslant 0. \tag{55}$$

Since $\vec{x_1}$, $\vec{x_2}$ and τ are arbitrary, we have

$$\eta_1(\vec{x},t)\leqslant 0, \tag{56}$$

$$\eta_2(\vec{x},t)\leqslant 0, \tag{57}$$

i.e.,

$$\left|\nabla T(\vec{x},t)\right|\leqslant\sqrt{\frac{M_1}{V}}\varepsilon,\quad \vec{x}\in\Omega, \tag{58}$$

$$\left|T(\vec{x},t)\right|\leqslant\sqrt{\frac{M_2}{S}}\varepsilon,\quad \vec{x}\in\partial\Omega. \tag{59}$$

Therefore, as $\varepsilon\to 0$,

$$\left|\nabla T(\vec{x},t)\right|\to 0\quad\forall\vec{x}\in\Omega, \tag{60}$$

and

$$\left|T(\vec{x},t)\right|\to 0\quad\forall\vec{x}\in\partial\Omega. \tag{61}$$

Let $\vec{x}$ be an arbitrary point in Ω, $\vec{x_0}$ a point on $\partial\Omega$. There exists a point $\vec{y}$ in Ω between $\vec{x}$ and $\vec{x_0}$, by the Lagrange mean-value theorem, such that

$$T(\vec{x},t) = T(\vec{x_0},t) + \nabla T(\vec{y},t)\cdot(\vec{x}-\vec{x_0}). \tag{62}$$

Therefore,

$$|T(\vec{x},t)|\leqslant|T(\vec{x_0},t)| + |\nabla T(\vec{y},t)||\vec{x}-\vec{x_0}|, \tag{63}$$

which, with Eqs. (60) and (61), leads to

$$|T(\vec{x},t)|\to 0,\ \forall\vec{x}\in\Omega,\quad \text{as } \varepsilon\to 0. \tag{64}$$

Eqs. (61) and (64) conclude that there exists a positive constant c such that

$$|T(\vec{x},t)|\leqslant c\varepsilon\quad\forall\vec{x}\in\Omega \text{ and } \forall\vec{x}\in\partial\Omega. \tag{65}$$

Therefore, the solution of (1) is stable with respect to the initial conditions.

6. Concluding remarks

The well-posedness is examined for n-dimensional dual-phase-lagging heat conduction equations with $n\geqslant 2$ under Dirichlet, Neumann or Robin boundary conditions. The method of separation of variables is used to find a solution. Inequality (22) developed in the present work is employed to establish its uniqueness and stability with respect to initial conditions. This is of fundamental importance for using dual-phase-lagging heat conduction equations in microscale heat conduction.

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References

[1] D.Y. Tzou, Macro- to Microscale Heat Transfer: The Lagging Behavior, Taylor & Francis, Washington, 1997.
[2] L. Wang, X. Zhou, Dual-Phase-Lagging Heat Conduction, Shandong University Press, Jinan, 2000 (ISBN 7-5607-2164-8, xvi + 475 pp).
[3] I.W. Boyd, Laser Processing of Thin Films and Microstructures, Springer, New York, 1989.
[4] G. Chryssolouris, Laser Machining, Theory and Practice, Springer, New York, 1991.
[5] J.A. Knapp, P. Børgesen, R.A. Zuhr, Beam–Solid Interactions: Physical Phenomena, vol. 157, Materials Research Society, Pittsburgh, 1990.
[6] D.E. Koshland, Engineering a small world from atomic manipulation to microfabrication, Science 254 (1991) 1300–1342.
[7] J. Narayan, V.P. Godbole, G.W. White, Laser method for synthesis and processing of continuous diamond films on nondiamound substrates, Science 52 (1991) 416–418.
[8] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1989) 41–73.
[9] R.A. Guyer, J.A. Krumhansi, Solution of the linearized Boltzmann equation, Phys. Rev. 148 (1966) 766–778.
[10] T.Q. Qiu, C.L. Tien, Heat transfer mechanisms during short-pulse laser heating of metals, J. Heat Transfer 115 (1993) 835–841.
[11] M.I. Kaganov, I.M. Lifshitz, M.V. Tanatarov, Relaxation between electrons and crystalline lattices, Sov. Phys. JETP 4 (1957) 173–178.
[12] S.I. Anisimòv, B.L. Kapeliovich, T.L. Perelman, Electron emisssion from metal surfaces exposed to ultra-short laser pulses, Sov. Phys. JETP 39 (1974) 375–377.
[13] D.Y. Tzou, A unified field approach for heat conduction from micro- to macro-scales, J. Heat Transfer 117 (1995) 8–16.
[14] D.Y. Tzou, Experimental support for the lagging response in heat propagation, AIAA J.Thermophys. 9 (1995) 686–693.
[15] D.Y. Tzou, Y.S. Zhang, An analytical study on the fast-transient process in small scales, Int. J. Eng. Sci. 33 (1995) 1449–1463.
[16] C.K. Lin, C.C. Hwang, Y.P. Chang, The unsteady solution of a unified heat conduction equation, Int. J. Heat Mass Transfer 40 (1997) 1716–1719.

[17] P. Antaki, Solution for non-Fourier dual phase lag heat conduction in a semi-infinite slab with surface heat flux, Int. J. Heat Mass Transfer 41 (1998) 2253–2258.
[18] D.W. Tang, N. Araki, Wavy, wavelike, diffusive thermal responses of finite rigid slabs to high-speed heating of laser-pulses, Int. J. Heat Mass Transfer 42 (1999) 855–860.
[19] M. Xu, L. Wang, Thermal oscillation and resonance in dual-phase-lagging heat conduction, Int. J. Heat Mass Transfer 45 (2002) 1055–1061.
[20] L. Wang, M. Xu, X. Zhou, Well-posedness and solution structure of dual-phase-lagging heat conduction, Int. J. Heat Mass Transfer 44 (2001) 1659–1669.
[21] D. Fournier, A.C. Boccara, Heterogeneous media and rough surfaces: a fractal approach for heat diffusion studies, Physica A 157 (1989) 587–592.